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ON UNIFORMITY OF SHELLS

M. EPSTEIN
Department of Mechanical Engineering, University of Calgary, 2500 University Drive NW. T2N 1N4, Calgary, Alberta, Canada
E-mail: epstein@enme.ucalgary.ca

M. DE LEÓN
Instituto de Matemáticas y Física Fundamental, Consejo Superior de Investigaciones Científicas, Serrano 123, 28006 Madrid, Spain
E-mail: mdeleon@pinar1.csic.es

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Abstract—Considering a shell as a genuine Cosserat surface leads naturally to a definition of material uniformity which entails an enlarged isotropy group operation. A strict homogeneity condition is obtained in terms of the vanishing of a surface inhomogeneity tensor and two director integrability conditions. © 1998 Elsevier Science Ltd.

1. INTRODUCTION

When can it be said that two points of a shell are made of the same material? In other words, how are the material responses of different points of a shell to be compared with each other? This question, which in the case of ordinary three-dimensional bodies has led to the theory of material uniformity (Noll, 1967; Wang, 1967), is complicated in the case of shells by two factors. The first of these is almost self-evident, arising as it does from the extra geometrical structure introduced by the embedding of a surface in space, particularly the curvature tensor. The second factor is of a more subtle nature and has to do with the very definition of a shell. Whether the theory of shells is derived through thickness-wise integration of the three-dimensional field equations, or identified *ab initio* with a Cosserat-type surface (Cohen and DeSilva, 1967), the outcome involves an embedded material surface endowed with an out-of-surface vector- (or *director-*)field. Accordingly, its deformations include both those of the surface and those of the vector field; but is this vector field part of the definition or, on the contrary, once a physical object is given which appears as a "surface with thickness", should we be free to choose the directors? In the latter case, of course, the theory should be invariant with respect to the choice of director field. Strong arguments support the adoption of this second point of view. Thus, if the theory is derived by integration, it is clear that the intention is to approximately represent the deformation of a three-dimensional element attached at each point of the surface as an affine deformation consistent with the deformation of the surface at the point of attachment. This affine deformation should be expressible in terms of *any* linearly independent triad.† One way to see the implications of the freedom of choice of the director field in a given reference configuration, is to compare two possible reference configurations with each other. If for instance, a thick plate originally in a configuration K_0 is given a pure transverse shear deformation so as to bring it to a new configuration K_1, there is no *a priori* reason to prefer K_0 to K_1 as reference configuration, so that, for example, the choice of the unit normal field should be admissible in either configuration. The case can be argued for the configuration with no initial stress, but such a configuration may, in general, not exist, not even locally.

† In their original work (Cosserat and Cosserat, 1909), the Cosserat brothers already adopted the point of view that the extra structure associated with the bending stiffness of a deformable surface is a point-wise affine deformation of a three-dimensional space, which they represented by means of the deformation of a triad attached at each point of the surface.

Adopting the more general point of view will have the effect of enlarging the material symmetry group of a shell point, since material automorphisms will be allowed which do not necessarily preserve the director or its derivatives. Correspondingly, the set of material isomorphisms between two points will be enlarged, as compared with that of a preferred-director approach. The missed-out symmetries may be important (one can think of a shell cut out from an elastic fluid, for example), or may be discarded *a posteriori* on physical grounds, but the fact remains that they cannot be altogether ignored when a shell is conceived as a surface carrying a point-wise three-dimensional structure, as physical reality itself suggests. An important work by Wang (1972) dealing with material uniformity of shells uses as a point of departure a definition of a shell as two deforming surfaces whose points are paired up. This ingenious device, while affording extra elegance in the formulation, is nevertheless equivalent to choosing a specific director field, unless a change of the pairing itself is allowed to be viewed as a change of reference configuration. A final point to be addressed is the definition of homogeneity. Whereas in the three-dimensional situation an unequivocal definition imposes itself naturally due to the existence of the background Euclidean parallelism, the case of a shell needs further consideration. An extreme point of view could be adopted, whereby the Euclidean parallelism still dictates the homogeneity condition. Accordingly, a shell would be locally homogeneous if each neighbourhood can be flattened to yield a plate in a homogeneous configuration, in the conventional sense. Other, less restrictive, definitions are possible [such as those of Ericksen (1970); Wang (1972)].

2. CONCEPTUAL FRAMEWORK

With an arbitrary frame chosen once and for all, physical space will, henceforth, be identified with $\mathbf{R}^3$, the collection of all real ordered triplets. In this space we consider an embedded surface $\mathbf{S}$, each point of which carries a copy of $\mathbf{R}^3$. Technically, this geometric entity is a *fibre bundle* $\mathbf{F}$, with *base manifold* $\mathbf{S}$ and typical fibre $\mathbf{R}^3$. The copy of $\mathbf{R}^3$ at the point p of $\mathbf{S}$ is called the *fibre at p*, denoted by $\mathbf{F}_p$.†

A *deformation K* of this fibre bundle consists of a smooth transformation onto a similar bundle, that is, a diffeomorphism κ of the surface $\mathbf{S}$ onto another embedded surface $\mathbf{s}$, plus a linear isomorphism of each of the fibres $\mathbf{F}_p$ onto its counterpart $\mathbf{f}_{\kappa(p)}$. This type of map between bundles is called a *fibre-bundle morphism*.

A long-standing tradition in shell theory advocates the use of *convected coordinates* to represent the various kinematic quantities associated with a deformation. In this article, at the expense of notational simplicity, we shall adhere to this tradition so that our results may be readily interpreted in familiar terms.‡ Let, then, a curvilinear coordinate system ξ^α ($\alpha = 1, 2$)§ be adopted on $\mathbf{S}$ and convected to all deformed surfaces $\mathbf{s}$ by the deformation κ itself. Physically, this means that material points preserve their coordinate values throughout the deformation process. Denoting by $\mathbf{R}$ and $\mathbf{r}$ the position vectors of points p and $\kappa(p)$, respectively in $\mathbf{S}$ and $\mathbf{s}$, and choosing bases $\mathbf{G}_i(\xi)$ and $\mathbf{g}_i(\xi)$ ($i = 1, 2, 3$) for the fibres at p and $\kappa(p)$, a deformation from the initial configuration with surface equation $\mathbf{R}(\xi)$ is described by the 12 functions

$$\xi^\alpha \to \mathbf{r}(\xi^\alpha), \quad H^j{}_i(\xi^\alpha) \tag{1}$$

where $\mathbf{H}$ is a point-dependent matrix whose physical meaning is as follows: any vector $\mathbf{v} \in \mathbf{F}_p$, with components v^i in the basis $\mathbf{G}_i$, is mapped onto the vector $K(\mathbf{v}) \in \mathbf{f}_{\kappa(p)}$ with components $H^j{}_i v^i$ in the basis $\mathbf{g}_j$. It is important to realize that, although in order to define $\mathbf{H}$ it would be sufficient to specify how one particular triad transforms, the true transformation is from the whole fibre to its counterpart, so that given $\mathbf{H}$ one can tell how any

† This is a very special fibre bundle. In a more general setting, S need not be an embedded surface, nor is the fibre at each p necessarily a mere copy of $\mathbf{R}^3$.

‡ The interested reader is referred to Epstein and de León (1996b, 1996c) for a rather more abstract treatment.

§ Greek indices will attain the values 1 and 2, while Latin indices will range between 1 and 3. The summation convention for diagonally repeated indices is adopted throughout.

spatial vector attached at a point of the surface transforms. This remark has important physical and mathematical consequences, particularly for the definition and study of inhomogeneities.

The *gradient of a deformation* will provide the deformed *natural base vectors* on **s**:

$$\mathbf{e}_\alpha = \frac{\partial \mathbf{r}}{\partial \xi^\alpha} \tag{2}$$

as well as information concerning the gradient of the fibre isomorphisms. Specifically, the derivative of κ at a point will be the two-point tensor

$$\mathbf{e}_\alpha \otimes \mathbf{E}^\alpha \tag{3}$$

where $\mathbf{E}^\alpha$ are the dual to the natural undeformed base vectors

$$\mathbf{E}_\alpha = \frac{\partial \mathbf{R}}{\partial \xi^\alpha} \tag{4}$$

and where indices are lowered (or raised) by means of the embedding-induced surface metrics

$$\begin{aligned} A_{\alpha\beta} &= \mathbf{E}_\alpha \cdot \mathbf{E}_\beta \\ a_{\alpha\beta} &= \mathbf{e}_\alpha \cdot \mathbf{e}_\beta \end{aligned} \tag{5}$$

for the reference and deformed surface metrics, respectively. A dot indicates the usual Cartesian inner product, available through the embedding. Similarly, the transformation between fibres is given by the two-point tensor

$$\mathbf{H} = H^j_{.i} \mathbf{g}_j \otimes \mathbf{G}^i \tag{6}$$

with indices lowered (or raised) by means of the (non-holonomic) metric matrices

$$\begin{aligned} G_{ij} &= \mathbf{G}_i \cdot \mathbf{G}_j \\ g_{ij} &= \mathbf{g}_i \cdot \mathbf{g}_j \end{aligned} \tag{7}$$

so that its (Euclidean) surface gradient must be carefully calculated as the third-order tensor

$$\nabla \mathbf{H} = \left(\frac{\partial H^j_{.i}}{\partial \xi^\alpha} - \Lambda^k_{i\alpha} H^j_{.k} + \lambda^j_{k\alpha} H^k_{.i} \right) \mathbf{g}_j \otimes \mathbf{G}^i \otimes \mathbf{E}^\alpha \tag{8}$$

where

$$\begin{aligned} \Lambda^i_{j\alpha} &= \frac{\partial \mathbf{G}_j}{\partial \xi^\alpha} \cdot \mathbf{G}^i \\ \lambda^i_{j\alpha} &= \frac{\partial \mathbf{g}_j}{\partial \xi^\alpha} \cdot \mathbf{g}^i . \end{aligned} \tag{9}$$

We now address the question of composition of maps. For this purpose we consider three configurations with position vectors $\mathbf{R}(\xi)$, $\mathbf{r}(\xi)$, $\hat{\mathbf{r}}(\xi)$, natural base vectors $\mathbf{E}_\alpha$, $\mathbf{e}_\alpha$, $\hat{\mathbf{e}}_\alpha$ and so on, respectively. The deformation from the first configuration to the second, and its derivative, are given by

$$\mathbf{r}(\xi^\alpha),\quad \mathbf{H}(\xi^\alpha) = H^j_{.i}\mathbf{g}_j \otimes \mathbf{G}^i$$

$$\mathbf{e}_\alpha \otimes \mathbf{E}^\alpha,\quad \nabla\mathbf{H} = \left(\frac{\partial H^j_{.i}}{\partial \xi^\alpha} - \Lambda^k_{i\alpha}H^j_{.k} + \lambda^j_{k\alpha}H^k_{.i}\right)\mathbf{g}_j \otimes \mathbf{G}^i \otimes \mathbf{E}^\alpha. \tag{10}$$

The counterparts for the deformation and derivative from the second configuration to the third are:

$$\hat{\mathbf{r}}(\xi^\alpha),\quad \mathbf{h}(\xi^\alpha) = h^j_{.i}\hat{\mathbf{g}}_j \otimes \mathbf{g}^i$$

$$\hat{\mathbf{e}}_\alpha \otimes \mathbf{e}^\alpha,\quad \nabla\mathbf{h} = \left(\frac{\partial h^j_{.i}}{\partial \xi^\alpha} - \lambda^k_{i\alpha}h^j_{.k} + \hat{\lambda}^j_{k\alpha}h^k_{.i}\right)\hat{\mathbf{g}}_j \otimes \mathbf{g}^i \otimes \mathbf{e}^\alpha. \tag{11}$$

Combining the two, we obtain the following formulas for the passage from the first configuration to the third:

$$\hat{\mathbf{r}}(\xi^\alpha), \mathbf{h}\circ\mathbf{H}(\xi^\alpha) = h^j_{.m}H^m_{.i}\hat{\mathbf{g}}_j \otimes \mathbf{G}^i, \hat{\mathbf{e}}_\alpha \otimes \mathbf{E}^\alpha$$

$$\nabla(\mathbf{h}\circ\mathbf{H}) = \left(\frac{\partial h^j_{.m}}{\partial \xi^\alpha}H^m_{.i} + h^j_{.m}\frac{\partial H^m_{.i}}{\partial \xi^\alpha} - \Lambda^k_{i\alpha}h^j_{.m}H^m_{.k} + \hat{\lambda}^j_{k\alpha}h^k_{.m}H^m_{.i}\right)\hat{\mathbf{g}}_j \otimes \mathbf{G}^i \otimes \mathbf{E}^\alpha. \tag{12}$$

The use of convected coordinates makes the underlying maps less transparent than they would appear in a general, coordinate free, notation, but the main fact remains true, namely, that the composition law for the derivative of the fibre isomorphism is less than trivial.

3. SHELLS

At this level of generality, the *structured surface* under consideration does not correspond to our intuitive grasp of a physical shell. This is so because there is no connection yet established between the deformation of the fibre and that of the tangent plane at the point of attachment to the surface. This connection can be established meaningfully if we restrict the fibre deformations to those which map the tangent planes onto the deformed tangent planes to the deformed surface at the target points. The important fact to bear in mind is that, even under that restriction, we are still mapping entire fibres to fibres, so that, although a single extra out-of-surface vector field would suffice to characterize this type of fibre deformation, we still have a (linear) rule to decide how *any other vector* in the fibre is deformed. This fact was noted by the brothers Cosserat in their original work (Cosserat and Cosserat, 1990). Physically, this means that a Cosserat surface is more than just a deforming surface with a deforming vector (or *director*) field. It is, rather, a deforming surface with a deforming three-dimensional space attached at each of its points. Mathematically, this implies that, even with the kinematic restriction imposed (preservation of tangent planes), the composition law for gradients still has to be observed. This remark will have repercussions upon the definition of the material symmetry group of a point in the shell.

Since we are considering surfaces embedded in $\mathbf{R}^3$, the unit normals $\mathbf{N}(\xi)$ and $\mathbf{n}(\xi)$ become available, respectively, in the reference and deformed configurations. Adopting $\mathbf{E}_\alpha$ and $\mathbf{N}$ as a fibre basis, our restricted (tangent-plane preserving) deformations imply that $\mathbf{H}$ is of the form:

$$\mathbf{H} = \mathbf{e}_\alpha \otimes \mathbf{E}^\alpha + u^\alpha\mathbf{e}_\alpha \otimes \mathbf{N} - u\mathbf{n} \otimes \mathbf{N} \tag{13}$$

which means that the out-of-surface part of $\mathbf{H}$ is completely characterized by a surface vector $u^\alpha\mathbf{e}_\alpha$ and a surface scalar u. Its gradient is:

$$\nabla \mathbf{H} = [(\gamma^{\alpha}_{\rho\beta} - \Gamma^{\alpha}_{\rho\beta} - u^{\alpha} B_{\rho\beta})\mathbf{e}_{\alpha} \otimes \mathbf{E}^{\rho} + (b_{\rho\beta} - uB_{\rho\beta})\mathbf{n} \otimes \mathbf{E}^{\rho} + (B^{\alpha}_{\beta} + u^{\alpha}_{;\beta} - ub^{\alpha}_{\beta})\mathbf{e}_{\alpha} \otimes \mathbf{N} + (u^{\alpha} b_{\alpha\beta} + u_{,\beta})\mathbf{n} \otimes \mathbf{N}] \otimes \mathbf{E}^{\beta} \quad (14)$$

where $B_{\alpha\beta}$ and $b_{\alpha\beta}$ are the curvature tensors, $\Gamma^{\alpha}_{\rho\beta}$ and $\gamma^{\alpha}_{\rho\beta}$ are the surface Christoffel symbols of **S** and **s**, respectively, and a comma and semicolon denote, respectively, ordinary and γ-covariant derivatives. The composition law applies just as before. Thus, for instance, the composition law for the fibre isomorphisms reads now:

$$\mathbf{h} \circ \mathbf{H} = \hat{\mathbf{e}}_{\alpha} \otimes \mathbf{E}^{\alpha} + (u^{\alpha} + \hat{u}^{\alpha} u)\hat{\mathbf{e}}_{\alpha} \otimes \mathbf{N} + \hat{u}u\hat{\mathbf{n}} \otimes \mathbf{N} \quad (15)$$

an apparently strange composition law, but one which simply expresses the successive application of two special linear maps between vector spaces. Note that this law allows us to compose *any* two such maps. The first one (**H**) takes the unit normal **N** into the vector $u^{\alpha}\mathbf{e}_{\alpha} + u\mathbf{n}$. The second map (**h**) carries the unit normal **n** into the vector $\hat{u}^{\alpha}\hat{\mathbf{e}}_{\alpha} + \hat{u}\hat{\mathbf{n}}$. The reason that these two maps can be composed at all is that they not only prescribe the deformation of the normals, but also (through the conservation-of-the-tangent-plane condition) they prescribe implicitly the deformation of any other spatial vector attached at the surface point.

For later use we consider now a particular kind of deformation. Let p and q be points in **S** and let $U, V \subset \mathbf{S}$ be open neighbourhoods of p and q, respectively, each with its own coordinate system, ξ^{α} and η^{α}. We wish to effect a deformation such that $\kappa(p) = q$ and $\kappa(U) \subset V$, namely, a deformation that maps a portion of the shell in the reference configuration into another such portion. It is clear that one way to describe a deformation of this type (on U) is to specify for each material point r in U with coordinates $\xi^{\alpha}(r)$, what are the values of the coordinates η^{α} of its target location $\kappa(r)$. In other words, we need two smooth functions

$$\eta^{\alpha} = \eta^{\alpha}(\xi^{\beta}) \quad (16)$$

satisfying

$$\eta^{\alpha}(\xi^{\beta}(p)) = \eta^{\alpha}(q). \quad (17)$$

In addition, we need a map of the unit normal fields, namely:

$$u^{\alpha} = u^{\alpha}(\xi^{\beta}), \quad u = u(\xi^{\beta}) \quad (18)$$

where these functions represent the components of the deformed unit normal in the triad made up of the natural base vector of η^{α} and the unit normal in V.†

Taking the derivative of the map (16), it follows that the base vectors $\mathbf{E}_{\alpha}(\xi)$ in U are mapped into the vectors

$$\frac{\partial \eta^{\alpha}}{\partial \xi^{\beta}} \mathbf{e}_{\alpha}(\eta) \quad (19)$$

where $\mathbf{e}_{\alpha}(\eta)$ are the natural base vectors of the η-coordinate system at the target points in V. Thus, the fibre-wise part of the deformation acquires now the form

$$\mathbf{P} = \eta^{\alpha}_{\beta}\mathbf{e}_{\alpha}(\eta(\xi)) \otimes \mathbf{E}^{\beta}(\xi) + u^{\alpha}\mathbf{e}_{\alpha}(\eta(\xi)) \otimes \mathbf{N}(\xi) + u\mathbf{n}(\eta(\xi)) \otimes \mathbf{N}(\xi)) \quad (20)$$

where we have notated:

† Alternatively, u^{α} may be construed as the components relative to the image of the natural base vectors of ξ^{α} by the deformation (16). The composition formulas will vary accordingly.

$$\eta^\alpha_\beta = \frac{\partial \eta^\alpha}{\partial \xi^\beta}. \tag{21}$$

The gradient of $\mathbf{P}$ at p is obtained, after some calculation, as

$$\nabla \mathbf{P} = (\eta^\beta_{\alpha\rho} + \eta^\mu_\alpha \eta^\sigma_\rho \gamma^\beta_{\mu\sigma} - \eta^\beta_\mu \Gamma^\mu_{\alpha\rho} - u^\beta B_{\alpha\rho})\mathbf{e}_\beta \otimes \mathbf{E}^\alpha \otimes \mathbf{E}^\rho + (\eta^\beta_\alpha \eta^\sigma_\rho b_{\beta\sigma} - uB_{\alpha\rho})\mathbf{n} \otimes \mathbf{E}^\alpha \otimes \mathbf{E}^\rho$$
$$+ \left(\eta^\beta_\alpha B^\alpha_\rho + \frac{\partial u^\beta}{\partial \xi^\rho} + u^\alpha \eta^\sigma_\rho \gamma^\beta_{\alpha\sigma} - u\eta^\sigma_\rho b^\beta_\sigma\right)\mathbf{e}_\beta \otimes \mathbf{N} \otimes \mathbf{E}^\rho + \left(u^\alpha \eta^\sigma_\rho b_{\alpha\sigma} + \frac{\partial u}{\partial \xi^\rho}\right)\mathbf{n} \otimes \mathbf{N} \otimes \mathbf{E}^\rho \tag{22}$$

with the notation

$$\eta^\beta_{\alpha\rho} = \frac{\partial^2 \eta^\beta}{\partial \xi^\rho \, \partial \xi^\alpha}. \tag{23}$$

It is noteworthy that the combination

$$\Delta^\alpha_{\beta\rho} = (\eta^{-1})^\alpha_\sigma \eta^\sigma_{\beta\rho} \tag{24}$$

transforms, under coordinate changes in U, as the Christoffel symbols of a symmetric connection.

An important particular case arises when $p = q$. We then have a local change of configuration given by

$$\mathbf{F}_1 = \eta^\alpha_\beta \mathbf{E}_\alpha(\eta(\xi)) \otimes \mathbf{E}^\beta(\xi) + u^\alpha(\xi)\mathbf{E}_\alpha(\eta(\xi)) \otimes \mathbf{N}(\xi) + u(\xi)\mathbf{N}(\eta(\xi)) \otimes \mathbf{N}(\xi) \tag{25}$$

with gradient at p given by

$$\nabla \mathbf{F}_1 = (\eta^\beta_{\alpha\rho} + \eta^\mu_\alpha \eta^\alpha_\rho \Gamma^\beta_{\mu\sigma} - \eta^\beta_\mu \Gamma^\mu_{\alpha\rho} - u^\beta B_{\alpha\rho})\mathbf{E}_\beta \otimes \mathbf{E}^\alpha \otimes \mathbf{E}^\rho + (\eta^\beta_\alpha \eta^\sigma_\rho B_{\beta\sigma} - uB_{\alpha\rho})\mathbf{N} \otimes \mathbf{E}^\alpha \otimes \mathbf{E}^\rho$$
$$+ \left(\eta^\beta_\alpha B^\alpha_\rho + \frac{\partial u^\beta}{\partial \xi^\rho} + u^\alpha \eta^\sigma_\rho \Gamma^\beta_{\alpha\sigma} - u\eta^\sigma_\rho B^\beta_\sigma\right)\mathbf{E}_\beta \otimes \mathbf{N} \otimes \mathbf{E}^\rho + \left(u^\alpha \eta^\sigma_\rho B_{\alpha\sigma} + \frac{\partial u}{\partial \xi^\rho}\right)\mathbf{N} \otimes \mathbf{N} \otimes \mathbf{E}^\rho. \tag{26}$$

Let now another such local change be induced by new functions $\zeta^\alpha(\xi^\beta)$, $v^\alpha(\xi^\beta)$, $v(\xi^\beta)$. The successive application of the two deformations corresponds then to the composition $\zeta^\alpha(\eta^\beta(\xi^\rho))$, $v^\alpha(\eta^\beta(\xi^\rho))$, $v(\eta^\beta(\xi^\rho))$. The fibre-wise part of the composition is now:

$$\mathbf{F}_2 \circ \mathbf{F}_1 = \zeta^\beta_\mu \eta^\mu_\alpha \mathbf{E}_\beta(\zeta(\eta(\xi))) \otimes \mathbf{E}^\alpha(\xi)$$
$$+ (u^\mu \zeta_m u^\beta + v^\beta u)\mathbf{E}_\beta(\zeta(\eta(\xi))) \otimes \mathbf{N}(\xi) + vu\mathbf{N}(\zeta(\eta(\xi))) \otimes \mathbf{N}(\xi) \tag{27}$$

and its gradient at p is

$$\nabla(\mathbf{F}_2 \circ \mathbf{F}_1) = [(\zeta^\alpha_{\lambda\mu}\eta^\mu_\rho \eta^\lambda_\beta + \zeta^\alpha_\lambda \eta^\lambda_{\beta\rho} + \zeta^\sigma_\lambda \eta^\lambda_\beta \zeta^\mu_\nu \eta^\nu_\rho \Gamma^\alpha_{\mu\sigma} - \zeta^\alpha_\lambda \eta^\lambda_\sigma \Gamma^\sigma_{\beta\rho} - (u^\lambda \zeta^\alpha_\lambda + v^\alpha u)B_{\beta\rho})\mathbf{E}_\alpha \otimes \mathbf{E}^\beta$$
$$+ (\zeta^\alpha_\lambda \eta^\lambda_\beta \zeta^\mu_\nu \eta^\nu_\rho B_{\alpha\mu} - vuB_{\beta\rho})\mathbf{N} \otimes \mathbf{E}^\beta + (\zeta^\alpha_\lambda \eta^\lambda_\sigma B^\sigma_\rho + u^\lambda_{,\rho}\zeta^\alpha_\lambda + u^\lambda \zeta^\alpha_{\lambda\sigma}\eta^\sigma_\rho + v^\alpha_{,\sigma}\eta^\sigma_\rho u + v^\alpha u_{,\rho}$$
$$+ (u^\lambda \zeta^\beta_\lambda + v^\beta u)\zeta^\sigma_\mu \eta^\mu_\rho \Gamma^\alpha_{\beta\sigma} - vu\zeta^\sigma_\mu \eta^\mu_\rho B^\alpha_\sigma)\mathbf{E}_\alpha \otimes \mathbf{N}$$
$$+ ((u^\lambda \zeta^\alpha_\lambda + v^\alpha u)\zeta^\mu_\nu \eta^\sigma_\rho B_{\alpha\mu} + \eta^\sigma_\rho v_{,\sigma} u + vu_{,\rho})\mathbf{N} \otimes \mathbf{N}] \otimes \mathbf{E}^\rho \tag{28}$$

admittedly a very complicated formula. Nevertheless, a comparison with the corresponding formula (22) for the gradient of one deformation alone, shows that this rather involved composition law can be stated symbolically as the comparatively simpler rule:

$$\begin{bmatrix} \zeta^\alpha_\beta \\ \zeta^\alpha_\sigma \\ v^\alpha \\ v^\alpha_{,\beta} \\ v \\ v_{,\beta} \end{bmatrix} \circ \begin{bmatrix} \eta^\alpha_\beta \\ \eta^\alpha_\sigma \\ u^\alpha \\ u^\alpha_{,\beta} \\ u \\ u_{,\beta} \end{bmatrix} = \begin{bmatrix} \zeta^\alpha_\sigma \eta^\sigma_\beta \\ \zeta^\alpha_{\sigma\phi}\eta^\phi_\rho + \eta^\sigma_\beta + \zeta^\alpha_\sigma \eta^\sigma_{\beta\rho} \\ u^\lambda \zeta^\alpha_\lambda + v^\alpha u \\ u^\lambda_{,\beta}\zeta^\alpha_\lambda + \eta^\sigma_\beta v^\alpha_{,\sigma} u + v^\alpha u_{,\beta} + \zeta^\alpha_{\lambda\sigma}\eta^\sigma_\beta u^\lambda \\ uv \\ u_{,\beta} v + u\eta^\phi_\beta v_{,\phi} \end{bmatrix}. \tag{29}$$

Finally, the composition of a special deformation **P** with an arbitrary one **h** of V yields, for the fibre-wise part the equation

$$\mathbf{h} \circ \mathbf{P} = \eta^\beta_\alpha \hat{\mathbf{e}}_\beta \otimes \mathbf{E}^\alpha + (u^\beta + \hat{u}^\beta u)\hat{\mathbf{e}}_\beta \otimes \mathbf{N} + \hat{u}u\hat{\mathbf{n}} \otimes \mathbf{N} \tag{30}$$

whose gradient a p is found to be:

$$\begin{aligned} \nabla(\mathbf{h} \circ \mathbf{P}) = [(&\eta^\beta_{\alpha\rho} + \eta^\mu_\alpha \eta^\sigma_\rho \hat{\gamma}^\beta_{\mu\sigma} - \eta^\beta_\sigma \Gamma^\sigma_{\alpha\rho} - (u^\beta + \hat{u}^\beta u)B_{\alpha\rho})\hat{\mathbf{e}}_\beta \otimes \mathbf{E}^\alpha \\ &+ (u^\beta_{,\rho} + \hat{u}^\beta_{,\sigma}\eta^\sigma_\rho u + \hat{u}^\beta u_{,\rho} - (u^\sigma + \hat{u}^\sigma u)\hat{\gamma}^\beta_{\sigma\mu}\eta^\mu_\rho - \hat{u}u\hat{b}^\beta_\sigma \eta^\sigma_\rho)\hat{\mathbf{e}}_\beta \otimes \mathbf{N} \\ &+ (\eta^\lambda_\alpha \eta^\sigma_\rho \hat{b}_{\lambda\sigma} - \hat{u}uB_{\alpha\rho})\hat{\mathbf{n}} \otimes \mathbf{E}^\alpha + ((u^\beta + \hat{u}^\beta u)\hat{b}_{\beta\sigma}\eta^\sigma_\rho + u_{,\rho}\hat{u} + u\hat{u})\hat{\mathbf{n}} \otimes \mathbf{N}] \otimes \mathbf{E}^\rho. \end{aligned} \tag{31}$$

4. CONSTITUTIVE LAWS AND MATERIAL UNIFORMITY

We turn to the question of material behaviour and, in particular, to the notion of material isomorphism. We restrict our attention to material responses which, at each point, depend at most on the values of the first gradient of the deformation. A (scalar) constitutive function will then be of the form

$$W = W(\nabla K; p) = W(\mathbf{H}, \nabla \mathbf{H}; p) \tag{32}$$

where **H** and its gradient ∇**H** are given, respectively, by eqns (13) and (14). The functional form of this equation depends, of course, on the reference configuration chosen. Note that because **H** already contains the information pertaining to the derivative of the position vector of the surface, the latter was not included, to avoid duplication.

At this point it appears appropriate to briefly consider the position of some variants of the theory of shells as used in applications. The first and most important is the Kirchhoff–Love theory (extended to the fully geometrically nonlinear range by Koiter (1966), Sanders (1963), and Budiansky (1968). In this theory one only admits normal-preserving deformations, ruling out transverse shear and transverse normal strains. In our notation, this implies that $u^\alpha = 0$ and $u = 1$ identically. The constitutive law reduces to:

$$\begin{aligned} W = W(&\mathbf{e}_\alpha \otimes \mathbf{E}^\alpha + \mathbf{n} \otimes \mathbf{N}, \\ &((\gamma^\alpha_{\rho\beta} - \Gamma^\alpha_{\rho\beta})\mathbf{e}_\alpha \otimes \mathbf{E}^\rho + (b_{\rho\beta} - B_{\rho\beta})\mathbf{n} \otimes \mathbf{E}^\rho + (B^\alpha_\beta - b^\alpha_\beta)\mathbf{e}_\alpha \otimes \mathbf{N}) \otimes \mathbf{E}^\rho ; \xi^\alpha) \end{aligned} \tag{33}$$

but since $\mathbf{E}_\beta$ and **N** are available, we may further restrict the theory by assuming that the dependence on the second tensorial argument is reduced by first applying it to $\mathbf{E}_\alpha$ and then to **N**. In this way, we effectively eliminate a second-order in-surface behaviour that would otherwise be included in the general first-order theory, due to the tangent-plane-preservation constraint. Applying further the first tensorial argument to $\mathbf{E}_\beta$, we obtain:

$$W = W(\mathbf{e}_\alpha, (B^\alpha_\beta - b^\alpha_\beta)\mathbf{e}_\alpha ; \xi^\alpha) \tag{34}$$

which, via the application of the *principle of frame indifference* yields the traditional form of the Kirchhoff–Love theory of shells:

$$W = W(a_{\alpha\beta}, b_{\alpha\beta}\,;\xi^{\alpha}). \tag{35}$$

Intermediate between the Kirchhoff–Love theory and the most general one, lies a theory with transverse shear and transverse stretch, but with a partial elimination of the second-order in-surface effects. It can be obtained by not assuming any restrictions on u^{α} or u and still applying the second tensorial argument first to $\mathbf{E}_{\alpha}$ and then to $\mathbf{N}$. Under such conditions we get:

$$W = W(\mathbf{e}_{\alpha} \otimes \mathbf{E}^{\alpha} + \mathbf{n} \otimes \mathbf{N}, ((B^{\alpha}_{\beta} + u^{\alpha}_{;\beta} - ub^{\alpha}_{\beta})\mathbf{e}_{\alpha} + (u^{\alpha}b_{\alpha\beta} + u_{,\beta})\mathbf{n})\,;\xi^{\alpha}) \tag{36}$$

which with the identification $\mathbf{d} = u^{\alpha}\mathbf{e}_{\alpha} + u\mathbf{n}$ is essentially the same as the elastic constitutive equation for a theory based on a single deformable director (Green *et al.*, 1965).

We focus our attention finally onto the problem of *material inhomogeneities.* Two points, p and q, are *materially isomorphic* if a deformation $\mathbf{P}$ (called a *material isomorphism*) of the special kind considered [eqn (20)] can be found for a neighbourhood of p into a neighbourhood of q, such that, after application of this deformation at p, the values of the constitutive quantity W at p and q become identical for all possible further deformations.† This condition is tantamount to finding fixed values

$$p^{\beta}_{\alpha}, p^{\beta}_{\alpha\rho} = p^{\beta}_{\rho\alpha}, w^{\alpha}, w^{\alpha}_{\beta}, z, z_{,\alpha} \tag{37}$$

for

$$\eta^{\beta}_{\alpha}, \eta^{\beta}_{\alpha\rho} = \eta^{\beta}_{\rho\alpha}, u^{\alpha}, u^{\alpha}_{\beta}, u, u_{,\alpha} \tag{38}$$

respectively, such that

$$W(\mathbf{h} \circ \mathbf{H}, \nabla(\mathbf{h} \circ \mathbf{H})(p)\,; p) = W(\mathbf{h}, \nabla\mathbf{h}(q)\,; q) \tag{39}$$

identically for all values of the arguments

$$\hat{\mathbf{e}}_{\alpha}, \hat{u}^{\alpha}, \hat{u}^{\alpha}_{\beta}, \hat{u}, \hat{u}_{\beta}, \hat{\gamma}^{\alpha}_{\rho\beta} = \hat{\gamma}^{\alpha}_{\beta\rho}, \hat{b}_{\rho\beta} = \hat{b}_{\beta\rho} \tag{40}$$

appearing in the definition of the arbitrary deformation $\mathbf{h}$. Notice that, because of the use of convected coordinates, the specification of these quantities is tantamount to choosing a deformation at points p and q with the same target. For the left-hand side of eqn (39), the expressions appearing in eqns (30) and (31) should be used. Following Noll (1967), we call a shell *materially uniform* if all its points are pairwise materially isomorphic. A *material symmetry* at a point is a material isomorphism between the point and itself. All material symmetries at a point form a group whose composition law is given by eqn (29).

It is interesting to note that even upon the reduction of eqn (36) to the single-deformable-director theory, the symmetry group is not reduced: It still abides by the general group operation (29) contrary to what is generally assumed (Wang, 1967; Carrol and Naghdi, 1972). If a fixed point q of the shell, together with a neighbourhood U and a fixed coordinate system x^{a} $(a = 1, 2)$‡ in U are adopted as a *reference crystal*, a field of material isomorphisms to all points of the shell will be given by functions

$$p^{a}_{\beta}(\xi), p^{a}_{\alpha\beta}(\xi) = p^{a}_{\beta\alpha}(\xi), w^{a}(\xi), w^{a}_{\alpha}(\xi), z(\xi), z_{\alpha}(\xi) \tag{41}$$

as $\xi(p)$ spans the shell. This *uniformity field* is in general not unique, since it is possible to compose it with a non-trivial point-dependent element of the symmetry group and still obtain a different uniformity field. We assume that the shell is *smoothly uniform* in the sense that it can be covered with patches in each of which the uniformity field can be chosen as

† We are tacitly assuming that W measures a quantity per unit mass. For a quantity per unit area a multiplicative correction by the appropriate metric determinant is needed.

‡ Here we taken exception to the rule that Roman indices range between 1 and 3.

a smooth function of position. Within the set of all possible smooth uniformity fields in a patch, there may be one satisfying an integrability or compatibility condition of some kind which is deemed to represent the physical notion of *homogeneity*. If that is the case, the shell is said to be homogeneous in the patch. A shell is *locally homogeneous* if it can be covered with coordinate patches in each of which the shell is homogeneous.

So as to elucidate what might the mathematical expression of the homogeneity condition be, we start by noticing that the field of matrices $p^a_\beta(p)$ can be seen as a (distant) *material parallelism* on the patch. Two vectors tangent to the surface at different points of the patch are materially parallel if their components are the same in their respective local *crystallographic bases*

$$\mathbf{f}_a = (p^{-1})^\beta_a \mathbf{e}_\beta. \tag{42}$$

Accordingly, the Christoffel symbols of this (local) *material connection* are given by:

$$\Lambda^\alpha_{\rho\beta} = -(p^{-1})^\alpha_{a,\beta} p^a_\rho. \tag{43}$$

This connection will, in general, have a non-vanishing *torsion tensor*

$$\tau^\alpha_{\beta\rho} = \Lambda^\alpha_{\beta\rho} - \Lambda^\alpha_{\rho\beta}. \tag{44}$$

We have already noticed [eqn (24)] that the quantities

$$\Delta^\alpha_{\rho\beta} = -p^a_{\rho\beta}(p^{-1})^\alpha_a \tag{45}$$

can themselves be viewed as the Christoffel symbols of a symmetric connection. We define now the *Cosserat surface inhomogeneity tensor* by means of the difference

$$D^\alpha_{\rho\beta} = \Lambda^\alpha_{\rho\beta} - \Delta^\alpha_{\rho\beta}. \tag{46}$$

The vanishing of this tensor on a patch implies, firstly, that the torsion τ of Λ vanishes. Thus, the crystallographic basis field becomes the natural basis of some curvilinear coordinate system in the patch. We conclude then that there exists a (not necessarily isometric) deformation of the patch into the plane that renders the distant material parallelism trivial (i.e. Euclidean). In this new configuration and (Cartesian) coordinate system, the Christoffel symbols of both Λ and Δ vanish identically in the patch. It follows from (43) that the new uniformity matrices p^a_α constitute a constant field and that $p^a_{\alpha\beta} = 0$. In effecting this possible change of configuration, we still have the degree-of-freedom to specify the fields u^α and u. We can always choose:

$$u^\alpha = (p^{-1})^\alpha_a w^a \tag{47}$$

and

$$u = z \tag{48}$$

and it is not hard to verify that the resulting new fields of uniformity for the director are

$$w^a = 0, \quad w^a_\rho = 0, \quad z = 1, \quad z_\rho = 0 \tag{49}$$

if, and only if,

$$z_\rho = \frac{\partial z}{\partial \xi^\rho} \tag{50}$$

and

$$w^a_\rho = \frac{\partial w^a}{\partial \xi^\rho} \tag{51}$$

throughout the patch. We conclude then that the vanishing of the Cosserat surface inhomogeneity tensor (for one possible uniformity matrix field) together with the integrability conditions (50) and (51) are equivalent to the most extreme form of homogeneity condition for the shell, namely, that the shell can be flattened by patches that yield a constitutive equation independent of position. It should be noted, finally, that due to the second-order effects, the homogeneity condition is still dependent to a certain extent on the reference crystal chosen. The reader is refered to Epstein and de León (1996a) for a discussion of this point.

4.1. *Remark*

It should be possible to extend the notion of shell uniformity in such a way that a shell-like structure cut out of a three-dimensional material be automatically uniform. In other words, rather than considering deformable surfaces in their own right, one should be able to subsume the two-dimensional formulation into a three-dimensional one whose uniformity and homogeneity can be determined by the ordinary theory of simple materials. Ideas of this kind have been proposed in Carrol and Naghdi (1972) and Naghdi and Rubin (1995) and, within the context of the geometrical theory of uniformity, will be the subject of a forthcoming article.

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